

# SHEAVES ON ABELIAN SURFACES AND STRANGE DUALITY

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**ABSTRACT.** We formulate three versions of a *strange duality* conjecture for sections of the Theta bundles on the moduli spaces of sheaves on abelian surfaces. As supporting evidence, we check the equality of dimensions on dual moduli spaces, answering a question raised by Göttsche-Nakajima-Yoshioka [GNY].

## 1. INTRODUCTION

Let  $(A, H)$  be a polarized abelian surface. In this paper, we consider the moduli spaces of Gieseker  $H$ -semistable sheaves on  $A$ , and sections of the Theta line bundles defined over them.

It will be convenient to bookkeep coherent sheaves  $E$  on  $A$  by their Mukai vectors, setting

$$v(E) = r + c_1(E) + \chi(E)\omega \in H^{2*}(A),$$

where  $\omega$  stands for the class of a point. As customary, we will equip the even cohomology  $H^{2*}(A)$  with the Mukai pairing. For any two vectors  $x = (x_0, x_2, x_4) \in H^{2*}(A)$  and  $y = (y_0, y_2, y_4) \in H^{2*}(A)$ , we set

$$\langle x, y \rangle = - \int_A x^\vee \cup y = \int_A (x_2 y_2 - x_0 y_4 - x_4 y_0).$$

It follows that for any two sheaves  $E$  and  $F$ , we have

$$\chi(E, F) = \sum_{i=0}^2 (-1)^i \operatorname{Ext}^i(E, F) = -\langle v(E), v(F) \rangle.$$

For an arbitrary  $v \in H^{2*}(A)$ , let us write  $\mathfrak{M}_v$  for the moduli space of Gieseker  $H$ -semistable sheaves  $E$  on  $A$ , with Mukai vector  $v$ . To keep things simple, we will make the following assumption throughout:

- Assumption 1.** (i) *The polarization  $H$  is generic i.e., it belongs to the complement of a locally finite union of hyperplane walls in the ample cone of  $A$ ;*  
(ii) *The vector  $v$  is a primitive element of the lattice  $H^{2*}(A, \mathbb{Z})$ ;*  
(iii) *The vector  $v$  is positive i.e., one of the following is true:*  
– *rank  $(v) > 0$ ;*  
– *rank  $(v) = 0$  and  $c_1(v)$  is effective,  $\chi(v) \neq 0$ , and  $\langle v, v \rangle \neq 0, 4$ .*

The choice of generic polarization will play only a minor role in what follows, and as such, we will suppress it from the notation. Note that (i) and (ii) together imply that  $\mathfrak{M}_v$

is a smooth manifold consisting of stable sheaves only. Its dimension equals  $2d_v + 2$ , with

$$(1) \quad d_v = \frac{1}{2} \langle v, v \rangle.$$

The main characters of our story will be a collection of naturally defined Theta line bundles on  $\mathfrak{M}_v$ . Consider the subgroup  $v^\perp$  inside the holomorphic  $K$ -theory of  $A$  generated by the sheaves  $F$  whose Mukai vectors  $w$  are orthogonal to  $v$ :

$$(2) \quad \chi(v \otimes w) = -\langle v^\vee, w \rangle = 0.$$

There is a morphism

$$\Theta : v^\perp \rightarrow \text{Pic}(\mathfrak{M}_v), \quad [F] \rightarrow \Theta_F,$$

constructed and studied by Li and Le Potier [Li] [LP] in the case of surfaces, and also by Drézet-Narasimhan in the case of curves [DN]. The construction is easiest to explain assuming that  $\mathfrak{M}_v$  is a fine moduli space, such that  $\mathcal{E}$  is the universal sheaf on  $\mathfrak{M}_v \times A$ . In this case, for a sheaf  $F$  with Mukai vector  $w$ , we set

$$(3) \quad \Theta_F = \det \mathbf{R}p_!(\mathcal{E} \otimes q^*F)^{-1},$$

where  $p$  and  $q$  are the two projections from  $\mathfrak{M}_v \times A$ . The orthogonality condition (2) is used to obtain a well defined line bundle  $\Theta_F$ , even in the absence of universal structures, by descent from the Quot scheme. Even though the line bundle  $\Theta_F$  depends on the  $K$ -theory class of  $F$ , the Chern class  $c_1(\Theta_F)$  depends only on the Mukai vector  $w$ . For simplicity of notation, in all cohomological computations below, we will write  $\Theta_w$  for any one of the line bundles  $\Theta_F$  as above. The finer dependence of the line bundles  $\Theta_F$  on the sheaf  $F$  will be discussed in more detail in Subsection 2.1.

Our goal in this paper is to compute the Euler characteristics of the line bundles  $\Theta_w$ , which can be interpreted as the  $K$ -theoretic Donaldson invariants of the abelian surface  $A$ . We will provide a simple expression for these Euler characteristics, valid in any rank. We will thus answer a question raised in [GNY], as part of a general study of the rank two  $K$ -theoretic Donaldson invariants of surfaces.

To explain the results, let us first fix a reference line bundle  $\Lambda$  on  $A$  with  $c_1(\Lambda) = c_1(v)$ . Then, we have a well-defined determinant morphism

$$(4) \quad \alpha_\Lambda^+ : \mathfrak{M}_v \rightarrow \widehat{A} = \text{Pic}^0(A), \quad E \rightarrow \det E \otimes \Lambda^{-1}.$$

Its fiber over the origin is the moduli space  $\mathbf{M}_v^+(\Lambda)$  of sheaves with fixed determinant  $\Lambda$ . The choice of the determinant is unimportant for our arguments, and therefore we will omit it from our notation when no confusion is likely to arise. We will show:

**Theorem 1.** *For any vectors  $v$  and  $w$  satisfying Assumption 1, and such that  $\chi(v \otimes w) = 0$ , we have*

$$(5) \quad \chi(\mathbf{M}_v^+, \Theta_w) = \chi(\mathbf{M}_w^+, \Theta_v) = \frac{1}{2} \frac{c_1(v \otimes w)^2}{d_v + d_w} \begin{pmatrix} d_v + d_w \\ d_v \end{pmatrix}.$$

There is yet another moduli space of interest to us, which is Fourier-Mukai ‘dual’ to the one considered above. Letting  $\mathcal{P}$  denote the normalized Poincaré bundle on  $A \times \widehat{A}$ , the Fourier-Mukai transform is defined by

$$(6) \quad \mathbf{R}\mathcal{S}(E) = \mathbf{R}p_!(\mathcal{P} \otimes q^*E) \in \mathbf{D}(\widehat{A}),$$

where  $p, q$  are the two projections. With this understood, let us set

$$\alpha_{\Lambda}^{-} : \mathfrak{M}_v \rightarrow A, \quad E \rightarrow \det \mathbf{RS}(E) \otimes \det \mathbf{RS}(\Lambda)^{-1}.$$

The fiber of the morphism  $\alpha_{\Lambda}^{-}$  over the origin is denoted by  $M_v^{-}$ , and parametrizes sheaves  $E$  with fixed determinant of the Fourier-Mukai transform. We will prove:

**Theorem 2.** *For any vectors  $v$  and  $w$  as in Theorem 1, we have*

$$(7) \quad \chi(M_v^{-}, \Theta_w) = \chi(M_w^{-}, \Theta_v) = \frac{1}{2} \frac{c_1(\hat{v} \otimes \hat{w})^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

Here  $\hat{v}$  and  $\hat{w}$  denote the Fourier-Mukai transforms of the two vectors  $v$  and  $w$ .

Finally, we may consider the morphism

$$a_v = (\alpha_{\Lambda}^{+}, \alpha_{\Lambda}^{-}) : \mathfrak{M}_v \rightarrow \hat{A} \times A.$$

This is the Albanese map of the moduli space  $\mathfrak{M}_v$ , cf. [Y1]. Its fiber over the origin will henceforth be denoted by  $K_v$ . We will show:

**Theorem 3.** *Assume that the Néron-Severi group of  $A$  has rank 1. With the same hypotheses as in Theorem 1, we have*

$$(8) \quad \chi(K_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

The manifest symmetry of the formulas in Theorems 1, 2 and 3 matches first of all that of their counterpart for the case of sheaves on a  $K3$  surface. Indeed, the theta Euler characteristics for the moduli space of sheaves on a  $K3$  were shown to be [GNY][OG]

$$(9) \quad \chi(\mathfrak{M}_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \binom{d_v + d_w + 2}{d_v + 1}.$$

This symmetry further suggests a general *strange duality* for surfaces, reminiscent of the case of moduli spaces of bundles on curves. There, the analogous invariance of the Verlinde formula reflects a geometric isomorphism between generalized theta functions with dual ranks and levels [MO1] [B]. In the case of sheaves on an abelian or  $K3$  surface, it is tempting to assert, similarly, that whenever defined and nonzero, the morphisms

$$SD^{\pm} : H^0(M_v^{\pm}, \Theta_w)^{\vee} \rightarrow H^0(M_w^{\pm}, \Theta_v)$$

are isomorphisms. The same considerations should apply to the companion morphism

$$SD : H^0(K_v, \Theta_w)^{\vee} \rightarrow H^0(\mathfrak{M}_w, \Theta_v).$$

In the above, for each of the three pairs of moduli spaces *i.e.*,  $(M_v^{\pm}, M_w^{\pm})$  and  $(K_v, \mathfrak{M}_w)$ , the line bundle  $\Theta_v$  stands for any one of the  $\Theta_E$ 's, for  $E$  in the corresponding moduli space of sheaves with Mukai vector  $v$ ; similarly,  $\Theta_w$  is any one of the line bundles  $\Theta_F$ , for  $F$  in the dual moduli space of sheaves with Mukai vector  $w$ . We will review the definition of the three *strange duality* morphisms in Section 2.1 below, assuming that

**Assumption 2.** *For any two (semi)-stable sheaves  $E$  and  $F$  with Mukai vectors  $v$  and  $w$ , we have*

$$H^2(E \otimes F) = 0.$$

*This is automatic if  $c_1(v \otimes w) \cdot H > 0$ , by Serre duality and stability.*

In order to use the numerics provided by Theorems 1, 2 and 3, one has to assume in addition that the line bundle  $\Theta_w$  has no higher cohomology on the various moduli spaces considered. The vanishing of higher cohomology is a delicate question, which can be answered satisfactorily only in few cases. For smooth moduli spaces, or for moduli spaces with mild singularities - *e.g.* rational - one may invoke standard vanishing theorems. These require the understanding of the positivity properties of  $\Theta_w$  *i.e.*, determining whether  $\Theta_w$  is big and nef. In the case under study, smoothness is assumed, bigness is easy to detect, and nefness is hoped for. This last point of nefness appears to be a subtle issue, even though the presence of the holomorphic symplectic structure on the moduli spaces considered here makes the question more tractable. A study for the Hilbert scheme of two points on  $K3$  surfaces and their deformations can be found for instance in [HT]. Nevertheless, Le Potier [LP] and Li [Li] proved the following results, which can be viewed as a higher dimensional generalization of the ampleness of the determinant line bundle on the moduli space of bundles over a curve.

- Fact 1.** (i) *If  $w$  has positive rank, and  $c_1(w)$  is a high multiple of the polarization  $H$ , then  $\Theta_w$  is relatively ample on the fibers of the determinant map  $\alpha^+$ .*
- (ii) *If  $w$  has rank 0, and  $c_1(w)$  is a positive multiple of the polarization, then  $\Theta_w$  is big and nef on the fibers of  $\alpha^+$ .*

Similar results should hold for the morphism  $\alpha^-$ . When the Picard rank of  $A$  is 1, this is obtained for free in many cases, by the remarks following Conjecture 2(ii).

One may then speculate

**Conjecture 1.** *When Assumptions 1 and 2 are satisfied, the three morphisms  $SD^+$ ,  $SD^-$  and  $SD$  are either isomorphisms or zero.*

This has the immediate

**Corollary 1.** *As  $E$  varies in  $K_v$ , the Theta sections  $\Theta_E$  on the dual moduli space  $\mathfrak{M}_w$  span the linear series  $|\Theta_v|$ . Same statements apply to the moduli spaces  $M_w^+$  and  $M_w^-$ , letting  $E$  vary in  $M_v^+$  and  $M_v^-$  respectively.*

The Conjecture was demonstrated in a number of cases, in this and other geometric setups. An overview of the already existing arguments, as well as proofs of new cases, can be found in [MO2].

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## 2. PRELIMINARIES ON THETA DIVISORS

**2.1. The strange duality morphisms.** We review here the definition of the *strange duality* morphisms. We will fix  $\Lambda$  an arbitrary determinant, and we let  $\hat{\Lambda} = \det \mathbf{RS}(\Lambda)$ . We recall the notation of the Introduction, letting  $M_v^+$  and  $M_v^-$  denote the moduli spaces of

sheaves with determinant  $\Lambda$  and determinant of the Fourier-Mukai transform equal to  $\widehat{\Lambda}$  respectively;  $K_v$  consists of sheaves satisfying both requirements.

We explained in the introduction that the line bundle  $\Theta_F$  only depends on the  $K$ -theory class of the reference sheaf  $F$ . We establish here the following more precise result.

**Lemma 1.** *Consider a sheaf  $F$  of Mukai vector  $w$ , and consider the line bundle  $\Theta_F$  on the moduli space  $\mathfrak{M}_v$ . Then,*

- (i) *for  $F \in \mathcal{M}_w^+$ , the restriction of  $\Theta_F$  to  $\mathcal{M}_v^+$  is independent of the choice of  $F$ ;*
- (ii) *for  $F \in \mathcal{M}_w^-$ , the restriction of  $\Theta_F$  to  $\mathcal{M}_v^-$  is independent of the choice of  $F$ ;*
- (iii) *for  $F \in \mathfrak{M}_w$ , the restriction of  $\Theta_F$  to  $K_v$  is independent of the choice of  $F$ .*

*Proof.* To prove (i), pick two sheaves  $F_1$  and  $F_2$  with Mukai vector  $w$  and the same determinant. Considering the virtual element in  $K$ -theory

$$f = F_1 - F_2,$$

we need to show that

$$\Theta_{F_1} \otimes \Theta_{F_2}^{-1} = \Theta_f$$

is trivial. We will show that in  $K$ -theory,

$$(10) \quad f = \mathcal{O}_Z - \mathcal{O}_W$$

for two zero-dimensional schemes  $Z$  and  $W$ , which necessarily have to be of the same length. This follows by induction on the rank of the  $F$ 's. The rank 0 case is obvious. When the rank is 1, then

$$F_1 = L \otimes I_Z, F_2 = L \otimes I_W$$

with  $L = \det F_1 = \det F_2$ , and the result is immediate. For the inductive step, note that it suffices to replace  $F_1$  and  $F_2$  by the twists  $F_1(D)$  and  $F_2(D)$ , for some ample divisor  $D$ . In this case, we reduce the rank by constructing exact sequences

$$0 \rightarrow \mathcal{O}_A \rightarrow F_i(D) \rightarrow F'_i \rightarrow 0$$

with  $F'_1$  and  $F'_2$  of the same lower rank and the same determinant. The claim then follows by the induction hypothesis applied to  $F'_1 - F'_2$ . Once (10) is understood, it suffices to assume that  $Z$  and  $W$  are supported on single points *i.e.*, that  $f$  is a formal sum

$$f = \sum_{z,w} (\mathcal{O}_z - \mathcal{O}_w).$$

In this case, we will check  $\Theta_f$  is trivial by testing against any  $S$ -family  $\mathcal{E} \rightarrow S \times A$  of sheaves with fixed determinant  $\Lambda$ . In particular, the latter requirement implies that

$$\det \mathcal{E} \cong M \boxtimes \Lambda$$

for some line bundle  $M$  on  $S$ . Then, the pullback of  $\Theta_f$  under the classifying morphism  $S \rightarrow \mathcal{M}_v^+$  is

$$\det p_!(\mathcal{E} \otimes q^* f)^{-1} = \left( \det \mathcal{E}|_{S \times \{z\}} \right)^{-1} \otimes \det \mathcal{E}|_{S \times \{w\}} \cong M^{-1} \otimes M \cong \mathcal{O}_S,$$

completing the proof of (i).

For (ii), the same reasoning applies to

$$\mathbf{RS}(f) = \mathbf{RS}(F_1) - \mathbf{RS}(F_2),$$

which can be assumed to be the difference of the structure sheaves of  $z$  and  $w$  on the dual abelian variety  $\widehat{A}$ . Therefore,

$$f = \mathcal{P}_z - \mathcal{P}_w,$$

where  $\mathcal{P}_z$  and  $\mathcal{P}_w$  are the line bundles on  $A$  represented by  $z$  and  $w$ . We need to check that

$$\det p_1(\mathcal{E} \otimes q^* \mathcal{P}_z) \cong \det p_1(\mathcal{E} \otimes q^* \mathcal{P}_w).$$

Consider the relative Fourier-Mukai sheaf on  $S \times \widehat{A}$

$$\det p_{13!}(p_{12}^* \mathcal{E} \otimes p_{23}^* \mathcal{P}) \cong M \boxtimes \widehat{\Lambda},$$

for some line bundle  $M$  on  $S$ . Here the pushforward and pullbacks are taken along the projections from  $S \times A \times \widehat{A}$ . The conclusion follows by looking at the isomorphic pullbacks of this sheaf over  $S \times \{z\}$  and  $S \times \{w\}$ .

Finally, the third statement is obvious since  $K_v$  is simply connected. Indeed, it suffices to check that  $c_1(\Theta_F)$  is independent of  $F$ . This is a Grothendieck-Riemann-Roch computation, using the defining formula (3), with  $\mathcal{E}$  replaced by a quasi universal sheaf, if needed.

**Remark 1.** The Lemma above is sufficient to our purposes. It may be useful to have a more detailed understanding of how the Theta line bundles  $\Theta_F$  vary with  $F$ . For the case of curves, the requisite formulas were established by Dr  zet-Narasimhan [DN]. We speculate that the following holds:

**Conjecture 2.** Consider two sheaves  $F_1$  and  $F_2$  with the same Mukai vector orthogonal to  $v$ .

(i) On  $M_v^+$ , we have

$$\Theta_{F_1} = \Theta_{F_2} \otimes (\alpha^-)^* (\det F_1 \otimes \det F_2^{-1}).$$

(ii) On  $M_v^-$ , we have

$$\Theta_{F_1} = \Theta_{F_2} \otimes ((-1) \circ \alpha^+)^* (\det \mathbf{RS}(F_1) \otimes \det \mathbf{RS}(F_2)^{-1}).$$

(iii) If  $c_1(v) = 0$ , then on  $\mathfrak{M}_v$  we have

$$\Theta_{F_1} = \Theta_{F_2} \otimes ((-1) \circ \alpha^+)^* (\det \mathbf{RS}(F_1) \otimes \det \mathbf{RS}(F_2)^{-1}) \otimes (\alpha^-)^* (\det F_1 \otimes \det F_2^{-1}).$$

Formula (i) is easily checked on the Hilbert scheme of points using that

$$\Theta_F = (\alpha^-)^* (\det F) \otimes E^{\text{rank } F},$$

where  $E$  is the exceptional divisor [EGL]. Assuming (i), evidence for (ii) is provided by the change of the Theta line bundles under Fourier-Mukai transform. Indeed, when the Picard number of  $A$  is 1, Yoshioka [Y1] exhibited very general examples of birational isomorphisms between the moduli spaces  $M_v^\pm$  and  $M_{\widehat{v}}^\mp$  on  $A$  and  $\widehat{A}$ , interchanging the maps  $\alpha^+$  and  $\alpha^-$ ; arguments of Maciocia [Ma] can be used to show that under this isomorphism the line bundle  $\Theta_F$  corresponds to  $\Theta_{(-1)^* \widehat{F}}$ , at least for generic  $F$  satisfying WIT. Finally item (iii) is consistent with (i) and (ii), and with Grothendieck-Riemann-Roch. It may be possible to prove all three formulas using suitable degeneration arguments.

Assuming Lemma 1, it is now standard to define the three *strange duality* morphisms. The construction is contained in [D] and [OG], but we will review it briefly here for the sake of completeness.

Recall that for any pair of sheaves  $(E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w$  we have

$$\chi(E \otimes F) = 0.$$

We will assume furthermore that

**Assumption 2.** (*modified version*)

- (a) either  $H^2(E \otimes F) = 0$ ; by stability this happens if  $c_1(E \otimes F).H > 0$ ;
- (b) or  $H^0(E \otimes F) = 0$ ; by stability this happens if  $c_1(E \otimes F).H < 0$ .

To treat all cases at once, let us denote by  $\mathcal{M}_v$  and  $\mathcal{M}_w$  any one of the three pairs  $(\mathcal{M}_v^+, \mathcal{M}_w^+)$ ,  $(\mathcal{M}_v^-, \mathcal{M}_w^-)$  and  $(K_v, \mathfrak{M}_w)$ . We set

$$(11) \quad \mathcal{L}_w = \begin{cases} \Theta_F, & \text{for } F \in \mathcal{M}_w, \text{ if Assumption 2 (a) holds,} \\ \Theta_F^{-1}, & \text{for } F \in \mathcal{M}_w, \text{ if Assumption 2 (b) holds.} \end{cases}$$

By Lemma 1, this is a well defined line bundle on  $\mathcal{M}_v$ . We similarly define the line bundle  $\mathcal{L}_v$  on  $\mathcal{M}_w$ .

Descent arguments, presented in detail in Dănilă's paper [D], show the existence of a natural divisor

$$\Delta_{v,w} \hookrightarrow \mathcal{M}_v \times \mathcal{M}_w$$

which is supported set-theoretically on the locus

$$\Delta_{v,w} = \{(E, F) \in \mathcal{M}_v \times \mathcal{M}_w, \text{ such that } h^1(E \otimes F) \neq 0\}.$$

This divisor is obtained as the vanishing locus of a section of a naturally defined line bundle  $\Theta_{v,w}$  on  $\mathcal{M}_v \times \mathcal{M}_w$ . The splitting

$$\Theta_{v,w} = \mathcal{L}_w \boxtimes \mathcal{L}_v$$

follows from Lemma 1 by the see-saw theorem. Therefore,  $\Delta_{v,w}$  becomes an element of

$$H^0(\mathcal{M}_v, \mathcal{L}_w) \otimes H^0(\mathcal{M}_w, \mathcal{L}_v)$$

inducing the duality morphism

$$H^0(\mathcal{M}_v, \mathcal{L}_w)^\vee \rightarrow H^0(\mathcal{M}_w, \mathcal{L}_v).$$

Note that when Assumption 2 (a) holds, the construction gives rise to the three duality morphisms of the Introduction:

$$(12) \quad \text{SD}^\pm : H^0(\mathcal{M}_v^\pm, \Theta_w)^\vee \rightarrow H^0(\mathcal{M}_w^\pm, \Theta_v), \text{ and}$$

$$(13) \quad \text{SD} : H^0(K_v, \Theta_w)^\vee \rightarrow H^0(\mathfrak{M}_w, \Theta_v).$$

### 3. EULER CHARACTERISTICS ON ALBANESE FIBERS

**3.1. The Albanese map.** In this section, we will compute the Euler characteristics of line bundles on  $K_v$ . We begin by reviewing a few facts about the Albanese map of  $\mathfrak{M}_v$ . Recall from the introduction the modified determinant morphism

$$\alpha_\Lambda^+ : \mathfrak{M}_v \rightarrow \widehat{A}, E \rightarrow \det E \otimes \Lambda^{-1},$$

and its Fourier-Mukai ‘dual’

$$\alpha_\Lambda^- : \mathfrak{M}_v \rightarrow A, E \rightarrow \det \mathbf{RS}(E) \otimes \widehat{\Lambda}^{-1}.$$

Putting these two morphisms together, we obtain the map

$$(14) \quad \mathbf{a}_v = (\alpha^+, \alpha^-) : \mathfrak{M}_v \rightarrow \widehat{A} \times A.$$

Yoshioka proved that  $\mathbf{a}_v$  is the Albanese map of the moduli space  $\mathfrak{M}_v$  [Y1].

The morphism (14) is easiest to understand for the vector  $v = (1, 0, n)$ . Then, the moduli space  $\mathfrak{M}_v$  is isomorphic to the product  $\widehat{A} \times A^{[n]}$  of the dual abelian variety and the Hilbert scheme  $A^{[n]}$  of points on  $A$ . The morphism  $\mathbf{a}_v$  can be identified with

$$1 \times s : \widehat{A} \times A^{[n]} \rightarrow \widehat{A} \times A$$

where the first map is the identity, while the second is induced by summation on the abelian surface. That is, for a zero-cycle  $Z$  supported on points  $z_i$  with length  $n_i$ , we let

$$s : A^{[n]} \rightarrow A, \quad s([Z]) = \sum_i n_i z_i.$$

The fiber of  $s$  over  $(0, 0)$  is the generalized Kummer variety  $K_{n-1}$  of dimension  $n - 1$ .

In general, Yoshioka studied the fiber of the Albanese map  $\mathbf{a}_v$  over the origin

$$K_v = \mathbf{a}_v^{-1}(0, 0),$$

under the assumption that the vector  $v$  is primitive and positive, in the sense discussed in the Introduction. In this situation, and when  $d_v \geq 3$ , Yoshioka proved that

- $K_v$  is an irreducible holomorphic symplectic manifold, deformation equivalent to the generalized Kummer surface  $K_{n-1}$ , with  $n = \langle v, v \rangle / 2$ .
- There is an isomorphism

$$H^2(K_v, \mathbb{Z}) \cong \text{Pic}(K_v) \cong v^\perp.$$

The vector  $w$  in  $v^\perp$  corresponds to the line bundle  $\Theta_w$  restricted to  $K_v$ .

- Moreover, if one endows  $H^2(K_v, \mathbb{Z})$  with the Beauville-Bogomolov form, and  $v^\perp$  with the intersection form, the above isomorphism is an isometry.

**3.2. Generalized Kummer varieties.** We start the calculation of Euler characteristics by considering the case of line bundles on the generalized Kummer varieties. In other words, we assume that  $v = (1, 0, n)$ .

For any divisor  $D$  on the abelian surface  $A$ , we let  $D_{(n)}$  be the divisor on  $A^{[n]}$  consisting of zero-cycles which intersect  $D$ . This divisor is a pull-back under the support morphism

$$f : A^{[n]} \rightarrow A^{(n)},$$



from the symmetric power  $A^{(n)}$  of  $A$ ,

$$D_{(n)} = f^*(D \boxtimes D \boxtimes \dots \boxtimes D)^{S_n}.$$

Further, let  $E$  be the exceptional divisor of  $A^{[n]}$  consisting of schemes with two coincident points in their support. Any line bundle on the Hilbert scheme is of the form  $D_{(n)} \otimes E^r$ . We will denote by the same symbol the restriction of these bundles to the generalized Kummer variety. We also do not distinguish notationally between line bundles and divisors.

**Lemma 2.**

$$\chi(K_{n-1}, D_{(n)} \otimes E^r) = n \binom{\chi(D) - (r^2 - 1)n - 1}{n - 1}$$

*Proof.* The expression given by the Lemma is a consequence of the known formula

$$(15) \quad \chi(A^{[n]}, D_{(n)} \otimes E^r) = \frac{\chi(D)}{n} \binom{\chi(D) - (r^2 - 1)n - 1}{n - 1},$$

which was deduced in [EGL]. To relate the two, we use the cartesian diagram

$$\begin{array}{ccc} K_{n-1} \times A & \xrightarrow{\sigma} & A^{[n]} \\ \downarrow p & & \downarrow s \\ A & \xrightarrow{n} & A \end{array}$$

The upper horizontal map is

$$\sigma : K_{n-1} \times A \rightarrow A^{[n]}, (Z, a) \mapsto t_a^* Z,$$

while the bottom morphism is the multiplication by  $n$  in the abelian surface. It follows that  $\sigma$  is an étale cover of degree  $n^4$ .

By the see-saw theorem, we find

$$\sigma^* D_{(n)} = D_{(n)} \boxtimes D^{\otimes n}, \text{ while } \sigma^* E = E \boxtimes \mathcal{O}_A.$$

Therefore

$$(16) \quad \chi(A^{[n]}, D_{(n)} \otimes E^r) = \frac{1}{n^4} \chi(K_{n-1}, D_{(n)} \otimes E^r) \chi(A, D^n) = \frac{\chi(D)}{n^2} \chi(K_{n-1}, D_{(n)} \otimes E^r).$$

Putting (15) and (16) together we obtain the Lemma.

**3.3. General Albanese fibers.** We can now consider the case of an arbitrary vector  $v$ . We claim

**Proposition 1.** *If  $d_v \neq 0$ , then*

$$\chi(K_v, \Theta_w) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

*Proof.* We prove here the statement when  $d_v \neq 2$ . The case  $d_v = 2$  will be considered separately in the next subsection.

When  $d_v = 1$  the Proposition follows immediately from Mukai and Yoshioka's results [Muk1], [Y2], as they have proved that the Albanese map  $a_v : \mathfrak{M}_v \rightarrow \widehat{A} \times A$  is an

isomorphism. Then  $K_v$  is a point, and both sides of the equation in Proposition 1 equal 1.

When  $d_v \geq 3$ , we follow the same arguments as in the case of  $K3$  surfaces. We will make use of the Beauville-Bogomolov form  $B$ . This quadratic form is defined on the second cohomology of any irreducible holomorphic symplectic manifold, and can be considered as a generalization of the intersection pairing on  $K3$  surfaces. In the case of the generalized Kummer varieties  $K_{n-1}$ , the form  $B$  gives an orthogonal decomposition

$$H^2(K_{n-1}, \mathbb{Z}) = H^2(A, \mathbb{Z}) \oplus \mathbb{Z}[E]$$

such that

$$B(c_1(D_{(n)})) = D^2, \quad B(c_1(E)) = -2n.$$

In particular,

$$B(c_1(D_{(n)} \otimes E^r)) = 2(\chi(D) - r^2 n).$$

Therefore, the result of Lemma 2 can be restated as

$$\chi(K_{n-1}, L) = n \binom{\frac{B(c_1(L))}{2} + n - 1}{n - 1},$$

for any line bundle  $L = D_{(n)} \otimes E^r$  on the Kummer variety  $K_{n-1}$ .

To get the result of the Proposition, we will use the fact that the Euler characteristics  $\chi(X, \mathcal{L})$  of any line bundle on an irreducible holomorphic symplectic manifold  $X$  can be expressed as a universal polynomial in the Beauville-Bogomolov form  $B(c_1(\mathcal{L}))$  [H] *i.e.*, a polynomial depending only on the underlying holomorphic symplectic manifold. This polynomial is an invariant of the deformation type. Moreover, the Beauville-Bogomolov form is also invariant under deformations. Now, by [Y1], we know that  $K_v$  is deformation equivalent to the generalized Kummer variety  $K_{n-1}$ , for  $n = \langle v, v \rangle / 2$ , and that

$$B(c_1(\Theta_w)) = \langle w, w \rangle.$$

Therefore,

$$\chi(K_v, \Theta_w) = d_v \binom{\frac{B(c_1(\Theta_w))}{2} + d_v - 1}{d_v - 1} = d_v \binom{d_v + d_w - 1}{d_v - 1} = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v},$$

completing the proof of the Proposition when  $d_v \neq 2$ .

**3.4. Two-dimensional Albanese fibers.** In this subsection, we establish Proposition 1 when  $d_v = 2$ , by analyzing the fairly involved special geometry of the situation. The case  $r = 1$  is covered by Lemma 2, so we assume that  $r \geq 2$ .  $K_v$  is now a  $K3$  surface. It suffices to show that

$$\chi(K_v, \Theta_w) = \frac{c_1(\Theta_w)^2}{2} + 2 = 2d_w + 2$$

or equivalently,

$$(17) \quad c_1(\Theta_w)^2 = 2\langle w, w \rangle.$$

Yoshioka identifies the  $K3$  surface  $K_v$  as a Fourier-Mukai partner of the Kummer surface  $X$  associated to  $A$  [Y1]. We review his construction below. To fix the notation, consider the following diagram

$$\begin{array}{ccccc}
\sqcup_{i=1}^{16} E_i & \hookrightarrow & \tilde{A} & \xrightarrow{p} & X & \longleftarrow & \sqcup_{i=1}^{16} C_i \\
& & \downarrow \pi & & & & \\
& & A & & & & 
\end{array}$$

Here  $\tilde{A}$  is the blowup of  $A$  at the 16 two-torsion points, with exceptional divisors denoted by  $E_i$ . Let

$$j : \bigsqcup_{i=1}^{16} E_i \rightarrow \tilde{A}$$

denote the inclusion of all 16 exceptional divisors in  $\tilde{A}$ . Finally,  $p$  is the morphism quotienting the  $\mathbb{Z}/2\mathbb{Z}$  automorphisms, and  $C_i$  are the  $(-2)$  curves on  $X$  which are the images of the exceptional divisors  $E_i$  under  $p$ .

Yoshioka proves that for a suitable isotropic Mukai vector  $\tau$  on  $X$  i.e.,  $\langle \tau, \tau \rangle = 0$ , and a suitable polarization  $L$ , the following isomorphism holds

$$(18) \quad K_v \cong \mathfrak{M}_X(\tau).$$

Here  $\mathfrak{M}_X(\tau)$  denotes the moduli space of  $L$ -semistable sheaves on  $X$ , with Mukai vector  $\tau$ . We will use Yoshioka's explicit isomorphism to identify the theta bundle  $\Theta_w$  on the moduli space  $\mathfrak{M}_X(\tau)$ .

We will explain the argument when  $r + c_1(v)$  is indivisible. The remaining case when  $r + c_1(v)$  equals twice a primitive class is entirely similar. Let us assume first that  $r \neq 2$ , and that either  $r$  and  $\chi(v)$  are both even, or  $r$  is odd. The isomorphism (18) associates to each  $F \in \mathfrak{M}_X(\tau)$  a sheaf  $E$  on  $A$ , via elementary modifications along the exceptional divisors on the blowup  $\tilde{A}$ . Concretely, the sheaf  $p^*F|_{E_i}$  splits as a sum

$$(19) \quad p^*F|_{E_i} \cong \mathcal{O}_{E_i}(-1)^{\oplus a_i} \oplus \mathcal{O}_{E_i}^{\oplus (r-a_i)},$$

for suitable integers  $a_i$ . Then,  $E$  is defined by the exact sequence

$$(20) \quad 0 \rightarrow \pi^*E \rightarrow p^*F \rightarrow j_* \left( \bigoplus_{i=1}^{16} \mathcal{O}_{E_i}(-1)^{\oplus a_i} \right) \rightarrow 0.$$

The assignment

$$\mathfrak{M}_X(\tau) \ni F \rightarrow E \in \mathfrak{M}_v,$$

establishes an isomorphism onto the image  $K_v$ .

In fact, Yoshioka's construction works in families, giving a natural transformation between the moduli functors

$$\underline{\mathfrak{M}}_X(\tau) \rightarrow \underline{K}_v.$$

The exact description of this transformation will be useful later. In what follows, let us agree that the base change of various morphisms to an arbitrary base  $S$  will be decorated by overlines. Fix any flat  $S$ -family  $\mathcal{F}$  of sheaves in  $\underline{\mathfrak{M}}_X(\tau)(S)$ . Define the sheaf  $\mathcal{G}$  on the union  $\bigsqcup_{i=1}^{16} E_i \times S$  via the exact sequence

$$(21) \quad 0 \rightarrow \bar{\pi}^* \bar{\pi}_* \bar{j}^* \bar{p}^* \mathcal{F} \rightarrow \bar{j}^* \bar{p}^* \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0.$$

The short exact sequence

$$(22) \quad 0 \rightarrow \bar{\pi}^* \mathcal{E} \rightarrow \bar{p}^* \mathcal{F} \rightarrow \bar{j}_* \mathcal{G} \rightarrow 0$$

then defines a new  $S$ -family  $\mathcal{E}$  in  $\underline{K}_v(S)$ .

Now let

$$(23) \quad \zeta = \text{ch}(p_!(\pi^*W))(1 + \omega) \in H^*(X),$$

be the Mukai vector of the pushforward  $p_!(\pi^*W)$ , for an arbitrary sheaf  $W$  on  $A$  with Mukai vector  $w$ . Using the exact sequence (20), and the fact that

$$\chi(j_*\mathcal{O}_{E_i}(-1) \otimes \pi^*w) = 0,$$

we find

$$2\chi(\tau \otimes \zeta) = \chi(p^*\tau \otimes \pi^*w) = \chi(\pi^*(v \otimes w)) = \chi(v \otimes w) = 0.$$

The above computation shows that  $\zeta$  defines a Theta line bundle  $\Theta_\zeta$  on  $\mathfrak{M}_X(\tau)$ . We claim that under the isomorphism  $K_v \cong \mathfrak{M}_X(\tau)$  we have an identification

$$(24) \quad \Theta_w \cong \Theta_\zeta.$$

As for any good quotient, the Picard group of the moduli scheme  $\mathfrak{M}_X(\tau)$  injects into that of the moduli functor  $\underline{\mathfrak{M}}_X(\tau)$ . Therefore, it suffices to check equality of the two line bundles  $\Theta_w$  and  $\Theta_\zeta$  over arbitrary base schemes  $S$ , and for arbitrary  $S$ -families  $\mathcal{F}$  of  $\underline{\mathfrak{M}}_X(\tau)$ .

Let  $q : S \times \tilde{A} \rightarrow S$  and  $\tilde{q} : S \times X \rightarrow S$  be the two projections. The exact sequence (22) and the push-pull formula then give

$$\Theta_w = \det \mathbf{R}q_!(\pi^*\mathcal{E} \otimes \text{pr}_A^*w)^{-1} = \det \mathbf{R}q_!(\bar{p}^*\mathcal{F} \otimes \text{pr}_A^*w)^{-1} = \det \mathbf{R}\tilde{q}_!(\mathcal{F} \otimes \text{pr}_X^*\zeta)^{-1} = \Theta_\zeta.$$

Here, we used that the contribution of the last term of (22) vanishes. Indeed, since

$$\bar{j}^*\text{pr}_A^*w = \text{rank}(w) \cdot 1,$$

we have

$$\det \mathbf{R}q_!(\bar{j}_*\mathcal{G} \otimes \text{pr}_A^*w) = \text{rank}(w) \cdot \det \mathbf{R}(q\bar{j})_!(\mathcal{G}) = 0.$$

The last equality follows from the fact that all direct images of  $\mathcal{G}$  vanish. This is implied by the base change theorem, observing that the restriction of  $\mathcal{G}$  to each fiber of the morphism  $q\bar{j} : S \times E_i \rightarrow S$  splits as a sum of line bundles  $\mathcal{O}_{E_i}(-1)$ . In turn this latter fact is a consequence of the defining exact sequence (22), in conjunction with equation (19).

To complete the proof, recall that Mukai [Muk3] established an isometric isomorphism

$$H^2(\mathfrak{M}_X(\tau)) \cong \tau^\perp / \tau$$

where the left hand side is endowed with the intersection pairing, while the right hand side carries the Mukai form induced from the cohomology  $H^*(X)$ . Then,

$$c_1(\Theta_w)^2 = c_1(\Theta_\zeta)^2 = \langle \zeta, \zeta \rangle = 2\langle w, w \rangle.$$

This proves (17).

The case  $r \neq 2$  and  $\chi(v)$  odd is entirely similar. In this case, the exact sequence (20) is replaced by

$$0 \rightarrow \pi^*E \rightarrow p^*F(E_1) \rightarrow j_*\left(\bigoplus_{i=1}^{16} \mathcal{O}_{E_i}(-1)^{\oplus a_i}\right) \rightarrow 0.$$

The argument identifying the Theta bundles carries through, for the vector

$$(25) \quad \zeta = \text{ch}(p_!(\pi^*W(E_1)))(1 + \omega).$$

The case  $r = 2$  requires a different discussion, since in this case, the description of the isomorphism (18) via the assignment  $F \rightarrow E$  is valid only on the complement of four rational curves  $R_i$ ,  $1 \leq i \leq 4$ . In fact, one cannot pick an isotropic vector  $\tau$  such that for each of the 16 exceptional divisors, the splitting type (19) is independent of the choice of a point  $[F] \in \mathfrak{M}_X(\tau)$ . At best, for a suitable  $\tau$ , the rigid splitting

$$p^*F|_{E_i} \cong \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(-1)$$

holds for 12 exceptional divisors  $E_i$ ,  $5 \leq i \leq 16$ . For the remaining four divisors  $E_i$ ,  $1 \leq i \leq 4$ , the splitting type varies within the moduli space.

When  $\chi(v)$  is even, for generic  $F$  in  $\mathfrak{M}_X(\tau)$ , the splitting type is

$$(26) \quad p^*F|_{E_i} = \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}, \quad 1 \leq i \leq 4.$$

The loci of non-generic splitting give rational curves  $R_i$  in  $K_v$ . Indeed, the nongeneric splitting is

$$(27) \quad p^*F|_{E_i} = \mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(1).$$

These  $F$ 's are shown to sit in exact sequences

$$(28) \quad 0 \rightarrow G_i \rightarrow F \rightarrow \mathcal{O}_{C_i}(-1) \rightarrow 0$$

for certain rigid stable bundles  $G_i$  on the  $K3$  surface  $X$ , cf. Lemma 4.23 in [Y1]. The nontrivial extensions (28) are parametrized by a rational curve

$$R_i \cong \mathbb{P}(\text{Ext}_X^1(\mathcal{O}_{C_i}(-1), G_i)).$$

When  $\chi(v)$  is odd, all these statements are true for the exceptional divisors  $E_2, E_3, E_4$ , but equations (26) and (27) fail for  $E_1$ . In fact, generically

$$(29) \quad p^*F|_{E_1} = \mathcal{O}_{E_1}(1) \oplus \mathcal{O}_{E_1}(1),$$

while nongenerically

$$(30) \quad p^*F|_{E_1} = \mathcal{O}_{E_1} \oplus \mathcal{O}_{E_1}(2).$$

The nongeneric splitting occurs along the rational curve

$$R_1 = \mathbb{P}(\text{Ext}^1(\mathcal{O}_{C_1}, G_1))$$

parametrizing extensions of the type

$$(31) \quad 0 \rightarrow G_1 \rightarrow F \rightarrow \mathcal{O}_{C_1} \rightarrow 0,$$

for some rigid vector bundle  $G_1$  on  $X$ .

We claim that the Theta bundles agree in this case as well *i.e.*, we check that the isomorphism (24)

$$\Theta_w \cong \Theta_\zeta$$

is satisfied. Let us first discuss the case when  $\chi(v)$  is even, with  $\zeta$  given by (23). To begin,  $\Theta_w$  and  $\Theta_\zeta$  agree on the complement of the four rational curves  $R_i$ ,  $1 \leq i \leq 4$ , since the exact sequence (20) is valid outside these curves. We will check that the Theta bundles agree along the curves  $R_i$  as well. Precisely, we claim that

$$(32) \quad c_1(\Theta_w) \cdot R_i = c_1(\Theta_\zeta) \cdot R_i = s, \quad 1 \leq i \leq 4,$$

with

$$s = \text{rank } w.$$

Moreover, the four curves  $R_i, 1 \leq i \leq 4$ , are disjoint. These facts will establish the isomorphism (24).

To calculate the first intersection in (32), we will use the explicit description of the rational curves  $R_i$  in the moduli space  $K_v$ . Specifically, Yoshioka notes that the curve  $R_i$  corresponds to those sheaves  $E$  on  $A$  which fail to be locally free at a two-torsion point  $x_i$ . We can construct these sheaves as elementary modifications of a fixed  $V$ :

$$(33) \quad 0 \rightarrow E \rightarrow V \rightarrow \mathcal{O}_{\{x_i\}} \rightarrow 0.$$

Since the middle sheaf  $V$  has Mukai vector  $v + \omega$ , it sits in a moduli space of dimension 2. Mukai showed that all such  $V$ 's are locally free [Muk3]. Therefore,  $E$  is not locally free at  $x_i$ , but it is locally free elsewhere. These nonlocally free elementary modifications are moreover parametrized by a  $\mathbb{P}^1$ , which should therefore be the rational curve  $R_i$  above. Moreover, the argument shows that the four rational curves  $R_i, 1 \leq i \leq 4$  are disjoint.

The universal structure on  $R_i \times A$ , associated to the elementary modifications (33), becomes

$$0 \rightarrow \mathcal{E} \rightarrow \mathrm{pr}_A^* V \rightarrow \mathcal{O}_{R_i}(1) \boxtimes \mathcal{O}_{\{x_i\}} \rightarrow 0.$$

Therefore,

$$c_1(\Theta_w) \cdot R_i = -c_1(p_!(\mathcal{E} \otimes \mathrm{pr}_A^* w)) = c_1(p_!(\mathcal{O}_{R_i}(1) \boxtimes (\mathcal{O}_{\{x_i\}} \otimes w))) = c_1(\mathcal{O}_{R_i}(1)^{\oplus s}) = s.$$

To prove the second equality of (32), we will use the description of the rational curves  $R_i$  provided by equation (28). The universal extension

$$0 \rightarrow \mathrm{pr}_X^* G_i \rightarrow \mathcal{F} \rightarrow \mathcal{O}_{R_i}(-1) \boxtimes \mathcal{O}_{C_i}(-1) \rightarrow 0$$

on  $R_i \times X$  restricts to (28) on the fibers of the projection  $p : R_i \times X \rightarrow R_i$ . Using this exact sequence, we compute

$$\begin{aligned} c_1(\Theta_\zeta) \cdot R_i &= -c_1(p_!(\mathcal{F} \otimes \mathrm{pr}_X^* \zeta)) = -c_1(p_!(\mathcal{O}_{R_i}(-1) \boxtimes (\mathcal{O}_{C_i}(-1) \otimes \zeta))) \\ &= -c_1(\mathcal{O}_{R_i}(-1)) \chi(\mathcal{O}_{C_i}(-1) \otimes \zeta) = c_1(\zeta) \cdot C_i = s. \end{aligned}$$

The last evaluation follows from (23) via Riemann-Roch.

When  $\chi(v)$  is odd, the numerics are slightly different, but (24) still holds for the vector  $\zeta$  given by (25). In this case, we check that

$$c_1(\Theta_\zeta) \cdot R_1 = s,$$

using equation (31), instead of (28).

This completes our analysis of the two-dimensional Albanese fibers, establishing Proposition 1.

#### 4. SHEAVES WITH FIXED DETERMINANT

In this section we will prove Theorem 1. We begin by fixing the notation. Specifically, let us write  $r, \Lambda, \chi$  for the rank, determinant and Euler characteristic of the vector  $v$ . The notation  $r', \Lambda', \chi'$  will be used for the vector  $w$ . The orthogonality of  $v$  and  $w$  translates into

$$(34) \quad r'\chi + c_1(\Lambda) \cdot c_1(\Lambda') + r\chi' = 0.$$

Let  $\mathcal{P}$  be the normalized Poincaré bundle on  $A \times \hat{A}$ . We make the convention that  $x$  will stand for a point of  $A$ , while  $y$  will be a point of  $\hat{A}$ . We will write

$$\mathcal{P}_x = \mathcal{P}|_{\{x\} \times \hat{A}}, \quad \mathcal{P}_y = \mathcal{P}|_{A \times \{y\}} \cong y.$$

We denote by  $t_x$  and  $t_y$  the translations by  $x$  and  $y$  on the abelian varieties  $A$  and  $\hat{A}$  respectively.

The following two facts about the Fourier-Mukai transform of an arbitrary  $E \in \mathbf{D}(A)$ , proved in [Muk1], will be used below:

$$(35) \quad \mathbf{RS}(E \otimes \mathcal{P}_y) = t_y^* \mathbf{RS}(E),$$

$$(36) \quad \mathbf{RS}(t_x^* E) = \mathbf{RS}(E) \otimes \mathcal{P}_{-x}.$$

It is moreover useful to recall that the two line bundles  $\Lambda$  and  $\hat{\Lambda}$  standardly induce morphisms

$$\begin{aligned} \Phi_\Lambda : A &\rightarrow \hat{A}, \quad x \mapsto t_x^* \Lambda \otimes \Lambda^{-1}, \text{ and} \\ \Phi_{\hat{\Lambda}} : \hat{A} &\rightarrow A, \quad y \mapsto t_y^* \hat{\Lambda} \otimes \hat{\Lambda}^{-1}, \end{aligned}$$

satisfying [Y1]

$$(37) \quad \Phi_\Lambda \circ \Phi_{\hat{\Lambda}} = -\chi(\Lambda)1, \quad \Phi_{\hat{\Lambda}} \circ \Phi_\Lambda = -\chi(\Lambda)1.$$

To start the proof of Theorem 1, consider the diagram

$$\begin{array}{ccc} K_v \times A & \xrightarrow{\Phi^+} & M_v^+ \\ \downarrow p & & \downarrow \alpha^- \\ A & \xrightarrow{\Psi^+} & A \end{array}$$

The upper horizontal map is given by

$$\Phi^+(E, x) = t_{rx}^* E \otimes t_x^* \Lambda^{-1} \otimes \Lambda.$$

This is well defined since

$$\det \Phi^+(E, x) = t_{rx}^* \Lambda \otimes (t_x \Lambda^{-1} \otimes \Lambda)^r = \Lambda.$$

**Lemma 3.** *The morphism  $\Psi^+$  is the multiplication by  $d_v$  in the abelian variety.*

*Proof.* Using (35) and (36), we compute

$$\begin{aligned} \alpha^- \circ \Phi^+(E, x) &= \det \mathbf{RS}(t_{rx}^* E \otimes t_x^* \Lambda^{-1} \otimes \Lambda) \otimes \hat{\Lambda}^{-1} = \det \mathbf{RS}(t_{rx}^* E \otimes \mathcal{P}_{-\Phi_\Lambda(x)}) \otimes \hat{\Lambda}^{-1} \\ &= \det \left( t_{-\Phi_\Lambda(x)}^* \mathbf{RS}(t_{rx}^* E) \right) \otimes \hat{\Lambda}^{-1} = t_{\Phi_\Lambda(-x)}^* \det \mathbf{RS}(t_{rx}^* E) \otimes \hat{\Lambda}^{-1} \\ &= t_{\Phi_\Lambda(-x)}^* \det (\mathbf{RS}(E) \otimes \mathcal{P}_{-rx}) \otimes \hat{\Lambda}^{-1} \\ &= t_{\Phi_\Lambda(-x)}^* (\det \mathbf{RS}(E) \otimes \mathcal{P}_{-rx}^\chi) \otimes \hat{\Lambda}^{-1} \\ &= t_{\Phi_\Lambda(-x)}^* \hat{\Lambda} \otimes \mathcal{P}_{-r\chi x} \otimes \hat{\Lambda}^{-1} = \Phi_{\hat{\Lambda}}(\Phi_\Lambda(-x)) \otimes \mathcal{P}_{-r\chi x} \\ &= \chi(\Lambda)x \otimes \mathcal{P}_{-r\chi x} = (\chi(\Lambda) - r\chi)x = d_v x. \end{aligned}$$

The first equality on the penultimate line follows from the fact that the Poincaré bundle  $\mathcal{P}_x$  is invariant under translations [M]. Equation (37) was used in the last line.

**Lemma 4.** *When  $d_v \neq 0$ , the diagram above is cartesian. Therefore, the morphism  $\Phi^+$  has degree  $d_v^4$ .*

*Proof.* This is almost immediate. Together,  $\Phi^+$  and  $p$  give rise to a morphism  $i : K_v \times A \rightarrow M_v^+ \times_{A, (\alpha^-, \Psi^+)} A$ . We show that  $i$  is an isomorphism. Since  $\Psi^+$  is étale, the natural morphism  $M_v^+ \times_{A, (\alpha^-, \Psi^+)} A \rightarrow M_v^+$  is also étale, so the fibered product  $M_v^+ \times_{A, (\alpha^-, \Psi^+)} A$  is smooth. The fibered product is also connected, as it follows by looking at the connected fibers of the projection to  $A$ ; note that the projection is surjective, as  $\alpha^-$  has this property, according to the previous Lemma. Therefore, it suffices to check that  $i$  is injective. If  $i(E, x) = i(E', x')$  then, by composing  $i$  with  $\Phi^+$  and  $p$ , we see that

$$t_{rx}^* E \otimes t_x^* \Lambda^{-1} \otimes \Lambda = t_{rx'}^* E' \otimes t_{x'}^* \Lambda^{-1} \otimes \Lambda, \text{ and } x = x'.$$

This immediately implies  $E = E'$  as well. The diagram is therefore cartesian.

**Proposition 2.** *We have*

$$(\Phi^+)^* \Theta_w \cong \Theta_w \boxtimes \mathcal{L}^+$$

where  $\mathcal{L}^+$  is a line bundle on  $A$  with

$$c_1(\mathcal{L}^+) = -d_v c_1(v \otimes w).$$

*Proof.* This follows by the see-saw theorem. Letting  $\Phi_x = \Phi^+|_{K_v \times \{x\}}$ , we claim that the pullback  $\Phi_x^* \Theta_w$  is independent of  $x$ , and therefore, by specializing to  $x = 0$ , it should coincide with  $\Theta_w$ . Since  $K_v$  is simply connected, it suffices to check that the Chern class  $c_1(\Phi_x^* \Theta_w)$  is independent of  $x$ . This is clear when a universal sheaf  $\mathcal{E}$  exists on  $M_v^+ \times A$ . Indeed, for  $F$  a sheaf on  $A$  with Mukai vector  $w$ ,

$$\Phi_x^* \Theta_w = \Phi_x^* (\det \mathbf{R}p_! (\mathcal{E} \otimes q^* F))^{-1} = (\det \mathbf{R}p_! ((1 \times t_{rx})^* \mathcal{E} \otimes q^* (t_x^* \Lambda^{-1} \otimes \Lambda \otimes F)))^{-1}.$$

The first Chern class can then be computed by Grothendieck-Riemann-Roch. The answer does not depend on the point  $x \in A$  since the maps  $(1 \times t_{rx})^*$  and  $t_x^*$  act as the cohomological restriction associated with  $K_v \times A \hookrightarrow M_v^+ \times A$ , and as the identity on the cohomology of  $A$ , respectively. When a universal family does not exist, one can use a quasi-universal family instead.

The above argument shows that  $(\Phi^+)^* \Theta_w$  should be of the form  $\Theta_w \boxtimes \mathcal{L}^+$  for some line bundle  $\mathcal{L}^+$  coming from  $A$ . We can express this line bundle explicitly as follows. Write

$$m = m_1 : A \times A \rightarrow A, (a, b) \rightarrow a + b$$

for the addition map, and consider the morphism

$$m_r : A \times A \rightarrow A, (a, b) \rightarrow a + rb.$$

Then,

$$m_r = m \circ (1, r).$$

Letting  $p_1, p_2$  be the two projections, we have

$$\mathcal{L}^+ = (\det \mathbf{R}p_{2!} (m_r^* E \otimes m^* \Lambda^{-1} \otimes p_1^* (\Lambda \otimes F)))^{-1}.$$

Letting  $\lambda = c_1(\Lambda)$ , we get by Grothendieck-Riemann-Roch,

$$c_1(\mathcal{L}^+) = -p_{2!} \left[ m_r^* v \cdot m^* e^{-\lambda} \cdot p_1^* (e^\lambda w) \right]_{(3)}.$$



Expanding each of the terms, we obtain

$$c_1(\mathcal{L}^+) = -p_{2!} \left[ (r + m_r^* \lambda + \chi m_r^* \omega) \cdot \left( 1 - m^* \lambda + \frac{\lambda^2}{2} m^* \omega \right) \cdot p_1^* \left( r' + (r' \lambda + \lambda') + \left( \chi' + \lambda \lambda' + r' \frac{\lambda^2}{2} \right) \omega \right) \right]_{(3)}.$$

The precise evaluation of this product relies on the following intersections

$$\begin{aligned} p_{2!}(m^* \lambda \cdot p_1^* \omega) &= \lambda, \quad p_{2!}(m_r^* \lambda \cdot p_1^* \omega) = r^2 \lambda, \\ p_{2!}(m_r^* \omega \cdot m^* \lambda) &= (r-1)^2 \lambda, \quad p_{2!}(m_r^* \lambda \cdot m^* \omega) = (r-1)^2 \lambda, \\ p_{2!}(m^* \omega \cdot p_1^* \alpha) &= \alpha, \quad p_{2!}(m_r^* \omega \cdot p_1^* \alpha) = r^2 \alpha, \text{ for any } \alpha \in H^2(A). \end{aligned}$$

The last pair of intersections is to be used for the class  $\alpha = r' \lambda + \lambda'$ . The formulas above are easily justified either by explicit computations in coordinates, or directly, by interpreting geometrically the intersections involved. For instance, the third pushforward  $p_{2!}(m_r^* \omega \cdot m^* \lambda)$  is computed as the image under  $p_2$  of the cycle

$$\{(a, b), a + rb = 0, a + b \in \lambda\} \hookrightarrow A \times A.$$

This pushforward can be identified with  $(r-1)^* \lambda = (r-1)^2 \lambda$ .

The value of the Chern class is obtained immediately from the previous intersections and a last one calculated by the Lemma below. Equation (34) has to be used to bring the answer in the form claimed by Proposition 2.

**Lemma 5.** *For any  $\lambda, \alpha \in H^2(A)$ , we have*

$$p_{2!}(m_r^* \lambda \cdot m^* \lambda \cdot p_1^* \alpha) = (r-1)^2 \left( \int_A \alpha \lambda \right) \cdot \lambda + r \lambda^2 \cdot \alpha.$$

*Proof.* First, note the isomorphism

$$m^* \Lambda \cong p_1^* \Lambda \otimes p_2^* \Lambda \otimes (1 \times \Phi_\Lambda)^* \mathcal{P}.$$

This shows that

$$(38) \quad m^* \lambda = p_1^* \lambda + p_2^* \lambda + (1 \times \Phi_\Lambda)^* c_1(\mathcal{P}), \text{ and}$$

$$m_r^* \lambda = (1 \times r)^* m^* \lambda = p_1^* \lambda + r^2 p_2^* \lambda + r \cdot (1 \times \Phi_\Lambda)^* c_1(\mathcal{P}).$$

It follows that

$$\begin{aligned} p_{2!}(m_r^* \lambda \cdot m^* \lambda \cdot p_1^* \alpha) &= (r^2 + 1) \left( \int_A \alpha \lambda \right) \cdot \lambda + r \cdot p_{2!}(p_1^* \alpha \cdot (1 \times \Phi_\Lambda)^* c_1(\mathcal{P})^2) \\ &= (r^2 + 1) \left( \int_A \alpha \lambda \right) \cdot \lambda + 2r \cdot \Phi_\Lambda^* \left\{ p_{2!} \left( p_1^* \alpha \cdot \frac{c_1(\mathcal{P})^2}{2} \right) \right\}. \end{aligned}$$

We will prove

$$(39) \quad \Phi_\Lambda^* \left\{ p_{2!} \left( p_1^* \alpha \cdot \frac{c_1(\mathcal{P})^2}{2} \right) \right\} = - \left( \int_A \alpha \lambda \right) \cdot \lambda + \frac{\lambda^2}{2} \cdot \alpha.$$

This follows by a computation in coordinates. Explicitly, let us write  $A = V/\Gamma$ . We regard  $V$  as a four-dimensional real vector space. The dual abelian variety has as underlying *real* manifold the torus  $V^\vee/\Gamma^\vee$ , where  $V^\vee$  stands for the *real* dual of  $V$ . Pick a basis  $f_1, f_2, f_3, f_4$  for  $V$ , which is symplectic for  $\Lambda$ . This means that in the dual basis,

$$\lambda = c_1(\Lambda) = d \cdot f_1^\vee \wedge f_2^\vee + e \cdot f_3^\vee \wedge f_4^\vee \in \Lambda^2 V^\vee,$$

for some (integers)  $d$  and  $e$ . Moreover, the Chern class of the Poincaré line bundle on  $A \times \widehat{A}$  takes the form

$$(40) \quad c_1(\mathcal{P}) = f_1^\vee \wedge f_1 + f_2^\vee \wedge f_2 + f_3^\vee \wedge f_3 + f_4^\vee \wedge f_4.$$

To prove (39), it suffices to assume that

$$\alpha = f_1^\vee \wedge f_2^\vee, \text{ or } \alpha = f_1^\vee \wedge f_3^\vee.$$

Let us consider only the first case, the second being similar. Then,

$$p_2! \left( p_1^* \alpha \cdot \frac{c_1(\mathcal{P})^2}{2} \right) = -f_3 \wedge f_4.$$

The discussion in [LB], chapter 2, and in particular Lemma 4.5 therein, shows that the map

$$\Phi_\Lambda^* : H^1(\widehat{A}, \mathbb{R}) \cong V \rightarrow H^1(A, \mathbb{R}) \cong V^\vee$$

is induced by the contraction of the first Chern class  $c_1(\Lambda)$ . It follows that

$$\Phi_\Lambda^* \left\{ p_2! \left( p_1^* \alpha \cdot \frac{c_1(\mathcal{P})^2}{2} \right) \right\} = -\Phi_\Lambda^* f_3 \wedge \Phi_\Lambda^* f_4 = -e^2 f_3^\vee \wedge f_4^\vee.$$

But this is also the result on the right hand side of (39):

$$- \left( \int_A \alpha \lambda \right) \cdot \lambda + \frac{\lambda^2}{2} \cdot \alpha = -e \cdot \lambda + de \cdot \alpha = -e^2 f_3^\vee \wedge f_4^\vee.$$

*Proof of Theorem 1.* When  $d_v \neq 0$ , Theorem 1 follows immediately from Propositions 1 and 2, and Lemma 4. Indeed, we have

$$\begin{aligned} \chi(\mathbf{M}_v^+, \Theta_w) &= \frac{1}{d_v^4} \chi((\Phi^+)^* \Theta_w) = \frac{1}{d_v^4} \chi(K_v, \Theta_w) \chi(A, \mathcal{L}^+) \\ &= \frac{1}{d_v^4} \cdot \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v} \cdot \frac{(d_v c_1(v \otimes w))^2}{2} = \frac{1}{2} \frac{c_1(v \otimes w)^2}{d_v + d_w} \binom{d_v + d_w}{d_v}. \end{aligned}$$

When  $d_v = 0$ , the Theorem is equivalent to the equality

$$\chi(\mathbf{M}_v^+, \Theta_w) = r^2.$$

It suffices to explain that the moduli space  $\mathbf{M}_v^+$  consists of  $r^2$  smooth points. By work of Mukai, it is known that  $\mathfrak{M}_v$  is an abelian surface. In fact, fixing  $E \in \mathbf{M}_v^+$ , we have an isogeny

$$A \rightarrow \mathfrak{M}_v, \quad x \rightarrow t_x^* E,$$

whose kernel is the group

$$K(E) = \{x \in A \text{ such that } t_x^* E \cong E\}.$$

Note that we may need to replace  $E$  by a twist  $E \otimes H^{\otimes n}$  to ensure that  $K(E)$  is finite. In this case,  $K(E)$  has  $\chi^2$  elements. This is a result of Mukai [Muk2]; to apply it, we

need to observe that  $E$  is a simple semi-homogeneous sheaf. Restricting to sheaves with determinant  $\Lambda$ , we see that

$$M_v^+ \cong K(\Lambda)/K(E),$$

has length  $\frac{\chi(\Lambda)^2}{\chi^2} = r^2$ .

## 5. SHEAVES WITH FIXED DETERMINANT OF THE FOURIER-MUKAI TRANSFORM.

This section is devoted to the proof of Theorem 2. It is possible to deduce this result from Theorem 1 when the Picard number of  $A$  is 1, by explicitly studying how the relevant moduli spaces and Theta divisors change under the Fourier-Mukai transform [Y1][Ma]. However, the following proof is simpler, covers all cases, and it is in the spirit of this paper. Note that the cohomological computation below may be regarded as the Fourier-Mukai 'dual' of last section's calculations.

We will crucially make use of the diagram

$$\begin{array}{ccc} K_v \times \hat{A} & \xrightarrow{\Phi^-} & M_v^- \\ \downarrow p & & \downarrow \alpha^+ \\ \hat{A} & \xrightarrow{\Psi^-} & \hat{A} \end{array}$$

The upper horizontal morphism  $\Phi^-$  is defined as

$$\Phi^-(E, y) = t_{\Phi_{\hat{\Lambda}}(y)}^* E \otimes y^\chi.$$

To check that  $\Phi^-$  is well defined, we compute

$$\begin{aligned} \det \mathbf{RS}(\Phi^-(E, y)) &= \det \mathbf{RS}(t_{\Phi_{\hat{\Lambda}}(y)}^* E \otimes y^\chi) = \det(t_{\chi y}^* \mathbf{RS}(E) \otimes \Phi_{\hat{\Lambda}}(y)^{-1}) \\ &= t_{\chi y}^* \det \mathbf{RS}(E) \otimes \Phi_{\hat{\Lambda}}(y)^{-\chi} = t_{\chi y}^* \hat{\Lambda} \otimes \Phi_{\hat{\Lambda}}(y)^{-\chi} \\ &= \hat{\Lambda} \otimes \Phi_{\hat{\Lambda}}(\chi y) \otimes \Phi_{\hat{\Lambda}}(y)^{-\chi} = \hat{\Lambda}. \end{aligned}$$

The next two results are the versions of Lemma 3 and Proposition 2 suitable to the present context.

**Lemma 6.** *The morphism  $\Psi^-$  is the multiplication by  $-d_v$  on the abelian variety  $\hat{A}$ .*

*Proof.* Using (36) and (37), we compute

$$\begin{aligned} \alpha^+ \circ \Phi^-(E, y) &= \det(t_{\Phi_{\hat{\Lambda}}(y)}^* E \otimes y^\chi) \otimes \Lambda^{-1} = t_{\Phi_{\hat{\Lambda}}(y)}^* \Lambda \otimes y^{r\chi} \otimes \Lambda^{-1} \\ &= \Phi_{\Lambda}(\Phi_{\hat{\Lambda}}(y)) \otimes y^{r\chi} = (-\chi(\Lambda) + r\chi)y = -d_v y. \end{aligned}$$

**Proposition 3.** *We have*

$$(\Phi^-)^* \Theta_w \cong \Theta_w \boxtimes \mathcal{L}^-,$$

where

$$c_1(\mathcal{L}^-) = -d_v c_1(\hat{v} \otimes \hat{w}).$$

*Proof.* The proof of this result parallels that of Proposition 2. It suffices to show that the line bundle  $\mathcal{L}^-$  corresponding to the divisor

$$\left\{ y \in \widehat{A}, \text{ with } H^0 \left( t_{\widehat{\Lambda}(y)}^* (E) \otimes y^\chi \otimes F \right) \neq 0 \right\}$$

has the first Chern class given by the Proposition. Note that

$$\mathcal{L}^- = (\det p_{2!} (f^* E \otimes p_1^* F \otimes \mathcal{P}^\chi))^{-1},$$

where

$$f : A \times \widehat{A} \rightarrow A \times A \rightarrow A$$

denotes the composition

$$(41) \quad f = m \circ (1 \times \Phi_{\widehat{\Lambda}}), \quad (x, y) \rightarrow x + \Phi_{\widehat{\Lambda}}(y).$$

By Riemann-Roch, we compute

$$c_1(\mathcal{L}^-) = -p_{2!} \left[ (r + f^* \lambda + \chi f^* \omega) \cdot (r' + p_1^* \lambda' + \chi' p_1^* \omega) \cdot \left( 1 + \chi c_1(\mathcal{P}) + \chi^2 \frac{c_1(\mathcal{P})^2}{2} \right) \right]_{(3)}.$$

The following observations allow for the explicit evaluation of the expression above:

$$p_{2!} \left( \frac{c_1(\mathcal{P})^2}{2} \cdot p_1^* \lambda' \right) = \widehat{\lambda}', \quad p_{2!} \left( \frac{c_1(\mathcal{P})^2}{2} \cdot f^* \lambda \right) = \widehat{\lambda},$$

$$(42) \quad p_{2!} (f^* \omega \cdot p_1^* \lambda') = \frac{\lambda^2}{2} \cdot \widehat{\lambda}' - (\lambda \cdot \lambda') \cdot \widehat{\lambda},$$

$$(43) \quad p_{2!} (f^* \lambda \cdot p_1^* \omega) = -\frac{\lambda^2}{2} \cdot \widehat{\lambda},$$

$$(44) \quad p_{2!} (f^* \omega \cdot c_1(\mathcal{P})) = -2\widehat{\lambda},$$

$$(45) \quad p_{2!} (f^* \lambda \cdot p_1^* \lambda' \cdot c_1(\mathcal{P})) = -\lambda^2 \cdot \widehat{\lambda}'.$$

The Proposition follows by substitution, also making straightforward use of the orthogonality constraint

$$r\chi' + \lambda \cdot \lambda' + r'\chi = 0.$$

It remains to explain the four numbered equations claimed above. Let us first consider (42). Interpreting the pushforward geometrically, and recalling the definition of  $f$  in (41), we find that

$$p_{2!} (f^* \omega \cdot p_1^* \lambda') = (-\Phi_{\widehat{\Lambda}})^* \lambda' = \Phi_{\widehat{\Lambda}}^* \lambda' = \frac{\lambda^2}{2} \cdot \widehat{\lambda}' - \left( \int_A \lambda \cdot \lambda' \right) \cdot \widehat{\lambda}.$$

The dual of the last equality was verified in (39). The case at hand is a corollary of what we have already shown there, using the fact that the Fourier-Mukai transform is an isometry. Equation (43) is very similar. To prove it, we observe that  $f$  restricts to  $\Phi_{\widehat{\Lambda}}$  on  $\{0\} \times \widehat{A}$ , hence

$$p_{2!} (f^* \lambda \cdot p_1^* \omega) = \Phi_{\widehat{\Lambda}}^* \lambda = -\frac{\lambda^2}{2} \cdot \widehat{\lambda}.$$

In turn, (44) follows by a computation in local coordinates. First, pick a basis  $f_1, f_2, f_3, f_4$  for  $V$  such that

$$c_1(\mathcal{P}) = f_1^\vee \wedge f_1 + f_2^\vee \wedge f_2 + f_3^\vee \wedge f_3 + f_4^\vee \wedge f_4.$$

From the definition of  $f$  in (41), we calculate

$$\begin{aligned} p_{2!}(f^* \omega \cdot c_1(\mathcal{P})) &= p_{2!}((1 \times \Phi_{\hat{\Lambda}})^* m^* \omega \cdot c_1(\mathcal{P})) = \\ &= -p_{2!} \left( (1 \times \Phi_{\hat{\Lambda}})^* \left( \sum_{j=1}^4 \text{PD}(f_j^\vee) \wedge f_j^\vee \right) \cdot \left( \sum_{j=1}^4 f_j^\vee \wedge f_j \right) \right) = \sum_{j=1}^4 \Phi_{\hat{\Lambda}}^* f_j^\vee \wedge f_j. \end{aligned}$$

Taking

$$\lambda = d \cdot f_1^\vee \wedge f_2^\vee + e \cdot f_3^\vee \wedge f_4^\vee,$$

this last expression is

$$(46) \quad 2d \cdot f_3 \wedge f_4 + 2e \cdot f_1 \wedge f_2 = -2\hat{\lambda},$$

confirming (44).

Finally, for (45), we observe that

$$\begin{aligned} p_{2!}(f^* \lambda \cdot p_1^* \lambda' \cdot c_1(\mathcal{P})) &= p_{2!}((1 \times \Phi_{\hat{\Lambda}})^* m^* \omega \cdot p_1^* \lambda' \cdot c_1(\mathcal{P})) \\ &= p_{2!}((1 \times \Phi_{\hat{\Lambda}})^* (1 \times \Phi_{\Lambda})^* c_1(\mathcal{P}) \cdot p_1^* \lambda' \cdot c_1(\mathcal{P})) \\ &= p_{2!}((1 \times (-\chi(\Lambda)))^* c_1(\mathcal{P}) \cdot p_1^* \lambda' \cdot c_1(\mathcal{P})) \\ &= -\chi(\Lambda) \cdot p_{2!}(c_1(\mathcal{P})^2 \cdot p_1^* \lambda') = \lambda^2 \cdot \hat{\lambda}'. \end{aligned}$$

The first line follows by the definition of  $f$  in (41), the second uses (38), while the third uses (37).

*Proof of Theorem 2.* As before, when  $d_v \neq 0$ , the Theorem follows immediately from Propositions 1 and 3, and Lemma 6. Using the cartesian diagram, we compute

$$\begin{aligned} \chi(\mathbf{M}_v^-, \Theta_w) &= \frac{1}{d_v^4} \chi((\Phi^-)^* \Theta_w) = \frac{1}{d_v^4} \chi(K_v, \Theta_w) \chi(A, \mathcal{L}^-) \\ &= \frac{1}{d_v^4} \cdot \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v} \cdot \frac{(d_v c_1(\hat{v} \otimes \hat{w}))^2}{2} = \frac{1}{2} \frac{c_1(\hat{v} \otimes \hat{w})^2}{d_v + d_w} \binom{d_v + d_w}{d_v}. \end{aligned}$$

When  $d_v = 0$ , we observe that  $\mathbf{M}_v^-$  consists of  $\chi^2$  smooth points. First, for any sheaf  $E$  in the moduli space  $\mathfrak{M}_v$ , consider the isogeny

$$\hat{A} \rightarrow \mathfrak{M}_v, y \mapsto E \otimes \mathcal{P}_y.$$

The kernel

$$\Sigma(E) = \{y \in \hat{A} \text{ such that } E \otimes \mathcal{P}_y \cong E\}$$

has length  $r^2$ , cf. [Muk2] (twisting by powers of  $H$  may be necessary). Note that the points in  $\mathbf{M}_v^-$  have the property

$$\det \mathbf{RS}(E \otimes \mathcal{P}_y) \otimes (\det \mathbf{RS} E)^{-1} \cong t_y^* \hat{\Lambda} \otimes \hat{\Lambda}^{-1} \cong \mathcal{O}.$$

Therefore,

$$\mathbf{M}_v^- = K(\hat{\Lambda})/\Sigma(E)$$

has length  $\chi(\hat{\Lambda})^2/r^2 = \chi^2$ , as claimed.

## 6. SHEAVES WITH ARBITRARY DETERMINANT.

This last section contains the proof of Theorem 3. The Euler characteristic on  $K_v$  was calculated in Proposition 1. To compute the one on  $\mathfrak{M}_w$ , we use the diagram

$$\begin{array}{ccc} K_w \times A \times \widehat{A} & \xrightarrow{\Phi} & \mathfrak{M}_w \\ \downarrow p & & \downarrow a_w \\ A \times \widehat{A} & \xrightarrow{\Psi} & A \times \widehat{A} \end{array} .$$

Here,  $\Phi : K_w \times A \times \widehat{A} \rightarrow \mathfrak{M}_w$  is defined as

$$\Phi(E, x, y) = t_x^* E \otimes y.$$

Using (35) and (36), Yoshioka proved in detail that

$$\Psi(x, y) = (-\chi'x + \Phi_{\widehat{\Lambda}}(y), \Phi_{\Lambda'}(x) + r'y),$$

which has degree  $d_w^4$  [Y1].

**Proposition 4.** *We have*

$$\Phi^* \Theta_v = \Theta_v \boxtimes \mathcal{L}$$

where  $\mathcal{L}$  is a line bundle on  $A \times \widehat{A}$  with

$$\chi(\mathcal{L}) = d_v^2 d_w^2.$$

*Proof.* It suffices to compute the Euler characteristic of the line bundle  $\mathcal{L}$  corresponding to the divisor

$$\{(x, y) \in A \times \widehat{A}, \text{ such that } H^0(t_x^* E \otimes y \otimes F) \neq 0\}.$$

In other words

$$\mathcal{L} = (\det p_{23!} (m_{12}^* E \otimes p_{13}^* \mathcal{P} \otimes p_1^* F))^{-1},$$

where the  $p$ 's denote the projections on the corresponding factors of  $A \times A \times \widehat{A}$ , while

$$m_{12} : A \times A \times \widehat{A} \rightarrow A$$

is the addition on the first two factors. Keeping the previous notations,

$$c_1(\mathcal{L}) = -p_{23!} \left[ (r + m_{12}^* \lambda + \chi m_{12}^* \omega) \cdot \left( 1 + p_{13}^* c_1(\mathcal{P}) + \frac{p_{13}^* c_1(\mathcal{P})^2}{2} \right) \cdot (r' + p_1^* \lambda' + \chi' p_1^* \omega) \right]_{(3)}.$$

Expanding, we easily obtain

$$-c_1(\mathcal{L}) = (\chi \lambda' + \chi' \lambda) + (r \widehat{\lambda}' + r' \widehat{\lambda}) - r \chi' c_1(\mathcal{P}) + p_{23!} (m_{12}^* \lambda \cdot p_{13}^* c_1(\mathcal{P}) \cdot p_1^* \lambda').$$

We claim that

$$\chi(\mathcal{L}) = \frac{c_1(\mathcal{L})^4}{4!} = d_v^2 d_w^2.$$

The computation makes use of the fact that the Picard number of  $A$  is 1, so we may assume that either  $\lambda' = 0$ , or otherwise that

$$\lambda = a \lambda'$$

for some constant  $a$ . In the first case, we have

$$c_1(\mathcal{L}) = -\chi' \lambda - r' \widehat{\lambda} + r \chi' c_1(\mathcal{P}).$$

To prove the claim, we first note that

$$(47) \quad \lambda \cdot \widehat{\lambda} \cdot \frac{c_1(\mathcal{P})^2}{2} = \lambda^2$$

This follows easily by a computation in local coordinates. Indeed, writing

$$(48) \quad \lambda = d f_1^\vee \wedge f_2^\vee + e f_3^\vee \wedge f_4^\vee,$$

and recalling that  $\widehat{\lambda}$  and  $c_1(\mathcal{P})$  have the form (46) and (40) respectively, we calculate

$$\lambda \cdot \widehat{\lambda} \cdot \frac{c_1(\mathcal{P})^2}{2} = 2de = \lambda^2.$$

With (47) understood, and using the fact that the Fourier-Mukai is an isometry, we obtain

$$\begin{aligned} \frac{c_1(\mathcal{L})^4}{4!} &= \frac{1}{4!} (\chi' \lambda + r' \widehat{\lambda} - r \chi' c_1(\mathcal{P}))^4 = \frac{r'^2 \chi'^2 (\lambda^2)^2}{4} + r' \chi' (r \chi')^2 \lambda^2 + (r \chi')^4 \\ &= \left( \frac{r' \chi' \lambda^2}{2} + (r \chi')^2 \right)^2 = \left[ r' \chi' \left( \frac{\lambda^2}{2} - r \chi \right) \right]^2 = d_v^2 d_w^2. \end{aligned}$$

The penultimate equality made use of the fact that  $r \chi' + r' \chi = 0$ .

Finally, the more general second case  $\lambda = a \lambda'$  is similar. Using (38), we get

$$p_{23!} (m_{12}^* \lambda \cdot p_{13}^* c_1(\mathcal{P}) \cdot p_1^* \lambda') = (\Phi_\Lambda \times 1)^* q_{23!} (q_{12}^* c_1(\mathcal{P}) \cdot q_{13}^* c_1(\mathcal{P}) \cdot q_1^* \lambda'),$$

with the  $q$ 's standing for the projections of the factors of  $A \times \widehat{A} \times \widehat{A}$ . In turn, we claim that

$$(49) \quad (\Phi_\Lambda \times 1)^* q_{23!} (q_{12}^* c_1(\mathcal{P}) \cdot q_{13}^* c_1(\mathcal{P}) \cdot q_1^* \lambda) = -\frac{\lambda^2}{2} c_1(\mathcal{P}).$$

Again, this is easiest to check in local coordinates. Assuming that (48) holds, we have

$$q_{23!} (q_{12}^* c_1(\mathcal{P}) \cdot q_{13}^* c_1(\mathcal{P}) \cdot q_1^* \lambda) = -df_3 \otimes f_4 + df_4 \otimes f_3 - ef_1 \otimes f_2 + ef_2 \otimes f_1.$$

Hence, after pullback by  $\Phi_\Lambda \times 1$ , the left hand side of (49) becomes

$$-de c_1(\mathcal{P}) = -\frac{\lambda^2}{2} c_1(\mathcal{P}),$$

as claimed. Putting things together, we obtain

$$c_1(\mathcal{L}) = -(\chi' a + \chi) \lambda' - (r' a + r) \widehat{\lambda}' + \left( r \chi' + \frac{a \lambda'^2}{2} \right) c_1(\mathcal{P}).$$

The same type of calculation as the one done above yields the answer

$$\chi(\mathcal{L}) = \frac{c_1(\mathcal{L})^4}{4!} = \left[ \frac{(\chi' a + \chi)(r' a + r) \lambda'^2}{2} + \left( r \chi' + \frac{a \lambda'^2}{2} \right)^2 \right]^2.$$

To conclude the proof, it remains to observe that the expression in square brackets can be equated with

$$-\left( \frac{a^2 \lambda'^2}{2} - r \chi \right) \left( \frac{\lambda'^2}{2} - r' \chi' \right) = -d_v d_w,$$

so that

$$\chi(\mathcal{L}) = (d_v d_w)^2.$$

The latter algebraic manipulation will be left to the reader, who may wish to use the fact that

$$a\lambda'^2 + r\chi' + r'\chi = 0.$$

It is very likely that the Lemma holds true for arbitrary abelian surfaces, without any restrictions on the Néron-Severi group, but the computation seems to be more involved.

*Proof of Theorem 3.* We compute

$$\begin{aligned} \chi(\mathfrak{M}_w, \Theta_v) &= \frac{1}{d_w^4} \chi(K_w, \Theta_v) \chi(A \times \hat{A}, \mathcal{L}) = \frac{1}{d_w^4} \cdot \frac{d_w^2}{d_v + d_w} \binom{d_v + d_w}{d_v} \cdot (d_v^2 d_w^2) \\ &= \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v} = \chi(K_v, \Theta_w). \end{aligned}$$

The case  $d_w = 0$  requires, as usual, special care. We need to show

$$\chi(\mathfrak{M}_w, \Theta_v) = d_v.$$

Using the degree  $\chi'^2$  isogeny:

$$\pi : A \rightarrow \mathfrak{M}_w, x \rightarrow t_x^* F,$$

where  $F$  is a semi-homogeneous sheaf of Mukai vector  $w$ , we have

$$\pi^* \Theta_v = (\det p_!(m^* F \otimes q^* E))^{-1},$$

with  $p, q$  and  $m$  standing for the projection and addition morphism. Then

$$c_1(\pi^* \Theta_v) = -\chi' \lambda - \chi \lambda'.$$

We obtain

$$\begin{aligned} \chi(\mathfrak{M}_w, \Theta_v) &= \frac{1}{\chi'^2} \chi(A, \pi^* \Theta_v) = \frac{1}{2\chi'^2} (\chi' \lambda + \chi \lambda')^2 = \frac{1}{2\chi'^2} (\chi'^2 \lambda^2 + \chi^2 \lambda'^2 - 2\chi \chi' (r\chi' + r'\chi)) \\ &= \frac{1}{\chi'^2} (\chi^2 d_w + \chi'^2 d_v) = d_v. \end{aligned}$$

This completes the proof of the Theorem.

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